

5 The One-Dimensional Wave Equation on the Line

5.1 Informal Derivation of the Wave Equation

We start here with a simple physical situation and derive the 1D wave equation. The hope is that this will provide you an initial intuitive feeling for expected behavior of solutions. It also gives importance to a fundamental equation, and gives physical meaning for initial, and boundary conditions. In a short appendix at the end of this section we present a couple other physical problems leading to the wave equation.

One of the first PDEs that was developed and worked on was a model of the vibrating string¹. Here is a rather informal derivation. Imagine a string lying entirely in the plane and along the x-axis. It is assumed struck or plucked and vibrates in the plane. Let $u = u(x, t)$ be the position of the (centerline of the) string at location x at time t (see figure 1). Assume the following:

1. Oscillations are small: $|u| \ll \text{length of the string } (= l)$;
2. Points on the string move vertically in the x, u plane;
3. Slope of the tangent to the string remains small: $|u_x| \ll 1$. So stretching of the string remains negligible. Arclength $\alpha = \alpha(t) = \int_0^l \sqrt{1 + (u_x)^2} dx \simeq l$;
4. The string is *perfectly flexible* (no resistance to bending); tension is in the tangent direction and horizontal tension is constant.

Consider a string segment $[x, x + \Delta x]$, $T(x, t)$ = tension at x at time t , $\Theta(x, t)$ = angle of string with respect to the x-axis at x at time t . By Newton's second law, $F(x, t) = \rho \Delta x \frac{\partial^2 u}{\partial t^2}$, where ρ is the linear density of the string (considered constant along the string), and the force comes from tension in the string only. There is no external force (such as gravity) exerted on the string. Horizontal tension is constant, and vertical tension moves the string vertically, so balancing forces in the horizontal direction gives

¹Jean d'Alembert, 1747. Jean le Rond d'Alembert (1717-1783), as a new born baby was left in a box on the steps of a church in Paris. He was placed in a foster home and in time became a leading intellectual giant in France, both in mathematics and in literature.

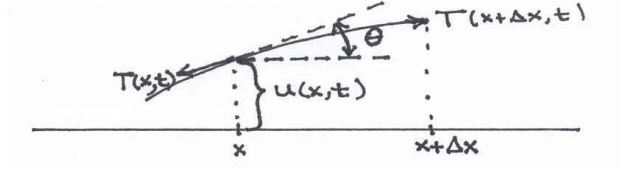


Figure 1: Notation for the derivation of the string equation

$T(x + \Delta x, t) \cos(\Theta(x + \Delta x, t)) = T(x, t) \cos(\Theta(x, t)) := \tau = \text{constant}$,
and in the vertical direction gives

$$\begin{aligned}
 F &= \rho \Delta x \frac{\partial^2 u}{\partial t^2}(\xi, t) \\
 &= T(x + \Delta x, t) \sin(\Theta(x + \Delta x, t)) - T(x, t) \sin(\Theta(x, t)) \\
 &= T(x + \Delta x, t) \cos(\Theta(x + \Delta x, t)) \tan(\Theta(x + \Delta x, t)) \\
 &\quad - T(x, t) \cos(\Theta(x, t)) \tan(\Theta(x, t)) \\
 &= \tau \left\{ \frac{\partial u}{\partial x}(x + \Delta x, t) - \frac{\partial u}{\partial x}(x, t) \right\}
 \end{aligned}$$

from Newton's second law, for some $\xi \in [x, x + \Delta x]$. Dividing this expression by $\rho \Delta x$ and letting $\Delta x \rightarrow 0$, we obtain

$$c^2 \frac{\partial^2 u}{\partial x^2}(x, t) = \frac{\partial^2 u}{\partial t^2}(x, t) , \quad (1)$$

where $c^2 = \tau/\rho > 0$.

Remark: Tension τ has units of *mass/time*², linear density has units of *mass/length*², so $c = \sqrt{\tau/\rho}$ has units of *length/time*; that is, it has units of *speed*, and we speak of c as the **wave speed**.

5.2 d'Alembert Solution

As d'Alembert did in the 18th century, consider a “really long” string; hence, let (1) hold for $|x| < \infty, t > 0$. We now do what is very common in PDEs, namely we transform the equation. Let

$$\xi = x - ct, \eta = x + ct, \text{ and } u(x, t) = v(\xi, \eta) .$$

This has the effect of rotating the coordinate axes so the equation collapses to a single derivative term that can be integrated directly. Mechanically, by the

chain rule, $u_x = v_\xi \frac{\partial \xi}{\partial x} + v_\eta \frac{\partial \eta}{\partial x} = v_\xi + v_\eta$. Also, $u_t = v_\xi \frac{\partial \xi}{\partial t} + v_\eta \frac{\partial \eta}{\partial t} = -cv_\xi + cv_\eta$. Taking derivatives again gives

$$u_{xx} = v_{\xi\xi} + 2v_{\xi\eta} + v_{\eta\eta}, \quad u_{tt} = c^2v_{\xi\xi} - 2c^2v_{\xi\eta} + c^2v_{\eta\eta},$$

which implies $u_{tt} - c^2u_{xx} = -4c^2v_{\xi\eta} = 0$, or $v_{\xi\eta} = 0$. Hence, $v_\xi = F'(\xi)$. Since the right hand side is an arbitrary function of the single variable ξ , it is convenient to write it as a derivative. Thus, $v(\xi, \eta) = F(\xi) + G(\eta)$. F, G are arbitrary differentiable functions of a single variable. Therefore,

$$u(x, t) = F(x - ct) + G(x + ct) . \quad (2)$$

This is the **general solution to the one-dimensional (1D) wave equation (1)**. We will return to giving an interpretation of (2) shortly.

The question arises why we chose $\xi = x - ct$, $\eta = x + ct$. First write (1) in the operator form:

$$0 = \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) u$$

and let $v = \left(\frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) u$. Hence,

$$0 = \frac{\partial v}{\partial t} + c \frac{\partial v}{\partial x} . \quad (3)$$

This is the simplest of first-order, linear PDEs we discussed in Section 4, and it can be thought of in a couple of ways. Mainly, it looks like a directional derivative. Note that for any differentiable function G of a single variable, $v(x, t) = G(x - ct)$ is a solution of (3): $\frac{\partial v}{\partial t} = -cG'(x - ct)$, $c \frac{\partial v}{\partial x} = cG'(x - ct)$. Hence, $v = \text{constant}$ along lines $x - ct = \text{constant}$ in the x, t plane. The set of lines $\{(x, t) : x - ct = \text{constant}\}$ is the family of **characteristics** (or **characteristic lines**) for the PDE (3), and these carry all the information for the solution. We could, having v , solve the first-order equation for u above, or we could have permuted the order of the first-order operators and arrived at

$$0 = \frac{\partial v}{\partial t} - c \frac{\partial v}{\partial x} \rightarrow v(x, t) = F(x + ct)$$

for any differentiable function F . In this case the set of characteristics is $\{(x, t) : x + ct = \text{constant}\}$. Therefore, the characteristics $x \pm ct = \text{constant}$, should play an integral part in the solution of the 1D wave equation, and suggests the transformation above.

Remark: Any solution $v(x, t) = G(x \pm ct)$ is called a **traveling wave solution**. Therefore, *the general solution, (2), of the wave equation, is the sum of a right-moving wave and a left-moving wave.*

5.3 The Cauchy Problem

Since (1) is defined on $|x| < \infty$, $t > 0$, we need to specify the initial displacement and velocity of the string. Let f, g be any functions defined on the real line that are piecewise continuous, and integrable (so they go to zero as $|x| \rightarrow \infty$). This last statement seems a bit technical, but we want to consider bounded solutions in this course, as much for physical reasons as for mathematical reasons). Therefore, assume

$$\begin{aligned} u(x, 0) &= f(x), \text{ for } |x| < \infty \\ \frac{\partial u}{\partial t}(x, 0) &= g(x) . \end{aligned} \tag{4}$$

The *Cauchy problem* for the wave equation is made up of (1) and (4). If we interpret u representing the vibration of a string, then we are specifying initial displacement and velocity in expressions (4) (as we must in ODE spring problems).

Note that, from (2), $f(x) = u(x, 0) = F(x) + G(x)$, which implies

$$cf'(x) = cF'(x) + cG'(x) . \tag{5}$$

(The primes here mean differentiation with regard to the single variable the function depends on.) Similarly, $g(x) = u_t(x, 0) = -cF'(x) + cG'(x)$. So adding this to (5), integrating, and dividing by $2c$ gives

$$G(x) = \frac{1}{2}f(x) + \frac{1}{2c} \int_{x_0}^x g(y)dy + C_1 . \tag{6}$$

Also, subtracting instead of adding gives

$$\begin{aligned} cf'(x) - g(x) &= 2cF'(x) \rightarrow F'(x) = \frac{1}{2}f'(x) - \frac{1}{2c}g(x) \rightarrow \\ F(x) &= \frac{1}{2}f(x) - \frac{1}{2c} \int_{x_0}^x g(y)dy + C_2 . \end{aligned} \tag{7}$$

Thus, putting the arguments used in (2) into (6),(7), we have

$$\begin{aligned} u(x, t) &= F(x - ct) + G(x + ct) = \frac{1}{2}\{f(x - ct) + f(x + ct)\} \\ &\quad - \frac{1}{2c} \int_{x_0}^{x-ct} g(y)dy + \frac{1}{2c} \int_{x_0}^{x+ct} g(y)dy + C_1 + C_2 \\ &= \frac{1}{2}\{f(x - ct) + f(x + ct)\} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)dy + C_1 + C_2 . \end{aligned}$$

But $u(x, 0) = f(x) = f(x) + C_1 + C_2$ implies $C_1 + C_2 = 0$, so finally

$$u(x, t) = \frac{1}{2}\{f(x - ct) + f(x + ct)\} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)dy . \quad (8)$$

This is **d'Alembert's formula**, or **d'Alembert's solution** to the Cauchy problem for the 1D wave equation on the line. Shortly we will give an interpretation of this solution form that will hopefully help you. But if a question calls for the general solution to the wave equation only, use (2). If the question involves (1) and initial data (4), then refer to (8).

Theorem: If f and g are continuous, with continuous derivatives $f'(x), f''(x), g'(x)$ on \mathbb{R} , then d'Alembert's formula (8) gives the (classical) solution to the wave equation (1) for all $x, t, |x| < \infty, t > 0$, and satisfies the initial conditions (4) on \mathbb{R} .

Example 1: Consider

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 && \text{on } |x| < \infty, t > 0 \\ u(x, 0) &= \frac{1}{10}e^{-x^2} && \text{on } |x| < \infty \\ u_t(x, 0) &= 0 && \text{on } |x| < \infty \end{aligned}$$

By (8), $u(x, t) = \frac{1}{20}\{e^{-(x-ct)^2} + e^{-(x+ct)^2}\} = \frac{1}{10}e^{-x^2-c^2t^2}\cosh(2cxt)$.

Remark: Figure 2 shows a plot of $u(x, t)$ for $c = 2$. We started with a “Gaussian” initial distribution that immediately breaks off into two “Gaussians”, only with smaller amplitude, each running off in different directions, but along straight lines. It appears as two diverging waves, which gives some justification to (1) being called the (1D) wave equation. The directions are the *characteristic directions* of the equation, here given by $x \pm ct = x \pm 2t = \text{constant}$.

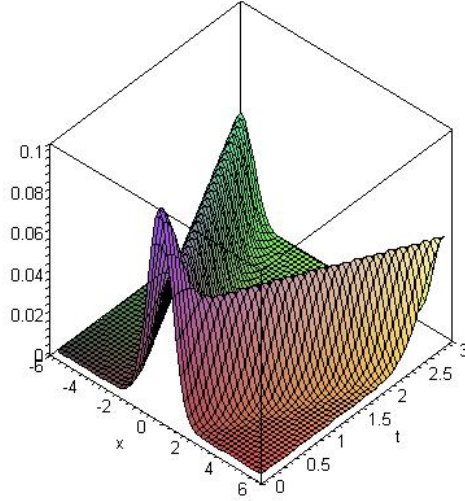


Figure 2: Graph of the solution to example 1

Example 2: Now consider

$$\begin{aligned} u_{tt} - u_{xx} &= 0 && \text{on } |x| < \infty, t > 0 \\ u(x, 0) &= \sin(x) && \text{on } |x| < \infty \\ u_t(x, 0) &= \cos(2x) && \text{on } |x| < \infty. \end{aligned}$$

By (8),

$$\begin{aligned} u(x, t) &= \frac{1}{2} \{ \sin(x - t) + \sin(x + t) \} + \frac{1}{2} \int_{x-t}^{x+t} \cos(2y) dy \\ &= \frac{1}{2} \{ \sin(x - t) + \sin(x + t) \} + \frac{1}{4} \{ \sin(2x + 2t) - \sin(2x - 2t) \} \\ &= \frac{1}{2} \{ \sin(x) \cos(t) - \cos(x) \sin(t) + \sin(x) \cos(t) + \cos(x) \sin(t) \} \\ &\quad + \frac{1}{4} \{ \sin(2x) \cos(2t) + \cos(2x) \sin(2t) - \sin(2x) \cos(2t) + \cos(2x) \sin(2t) \} \\ &= \sin(x) \cos(t) + \frac{1}{2} \cos(2x) \sin(2t) \\ &= \cos(t) \{ \sin(x) + \cos(2x) \sin(t) \} \end{aligned}$$

or some equivalent form using the trig addition formulas.

Exercises

1. Show that the solution to

$$\begin{aligned} u_{tt} - 4u_{xx} &= 0, \quad |x| < \infty, \quad t > 0 \\ u(x, 0) &= u_t(x, 0) = 2 \sin(3x) \end{aligned}$$

is given by $u(x, t) = 2 \sin(3x) \cos(6t) + \frac{1}{3} \sin(3x) \sin(6t)$.

2. Solve

$$\begin{aligned} u_{tt} &= 4u_{xx} \text{ on } |x| < \infty, \quad t > 0 \\ u(x, 0) &= 0 \text{ on } |x| < \infty \\ u_t(x, 0) &= \frac{1}{1+x^2/4} \text{ on } |x| < \infty \end{aligned}$$

(Answer: $u = \frac{1}{2}(\tan^{-1}(\frac{x+2t}{2}) - \tan^{-1}(\frac{x-2t}{2}))$.)

3. Verify that $u(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} g(y) dy$ is a solution to the wave equation $u_{tt} = c^2 u_{xx}$, satisfying initial data $u(x, 0) = 0, \frac{\partial u}{\partial t}(x, 0) = g(x)$.

4. Obtain the general solution to equation $u_{xx} - 3u_{xt} - 4u_{tt} = 0$. (Hint: first note that the equation is hyperbolic, so it has two real characteristics. For obtaining the analogue of (2), think of the operator split like $(\frac{\partial}{\partial x} - \alpha \frac{\partial}{\partial t})(\frac{\partial}{\partial x} + \beta \frac{\partial}{\partial t})u = 0$ and determine an appropriate set of values for α, β .)

(Answer: $u(x, t) = F(4x + t) + G(x - t)$, or equivalently, $u(x, t) = F(x + t/4) + G(x - t)$.)

5. For the Cauchy problem

$$\begin{aligned} u_{xx} + u_{xt} - 20u_{tt} &= 0 && \text{on } |x| < \infty, \quad t > 0 \\ u(x, 0) &= f(x), u_t(x, 0) = g(x) && \text{on } |x| < \infty \end{aligned}$$

first determine the general solution to the equation as done in the previous problem, then use the initial data to obtain an analogue to d'Alembert's formula.

(Answer: $u(x, t) = \frac{1}{9} \{4f(x + t/4) + 5f(x - t/5)\} + \frac{20}{9} \int_{x-t/5}^{x+t/4} g(y) dy$.)

6. What would be the analogue to d'Alembert's formula for the Klein-Gordon equation $u_{tt} - c^2 u_{xx} + m^2 u = 0, |x| < \infty, u(x, 0) = f(x), u_t(x, 0) = g(x)$? (Comment: This equation arises in quantum mechanics and will come up again in Section 7.)

7. Notice that we have specified data on the x -axis, which is transverse to the wave equation's characteristics. What if data is specified on a characteristic? This becomes a more delicate problem, but consider $u_{tt} - u_{xx} = 0$ with $u(x, 0) = f(x)$ and $u(x, x) = g(x)$. Since the general solution to the equation is $u(x, t) = F(x - t) + G(x + t)$, show that $G(x) = g(x/2) - F(0)$, $F(x) = F(0) + f(x) - g(x/2)$, and so $u(x, t) = f(x - t) + g(\frac{1}{2}(x + t)) - g(\frac{1}{2}(x - t))$. What is the consistency condition we would need for this to be a classical solution, assuming $f, g \in C^2(\mathbb{R})$?

Let us consider a few rather illustrative examples that violates the smoothness conditions of the theorem, but are easy to work with and helps us explain more about the wave equation (and hyperbolic equations in general).

Example 3: Consider the Cauchy problem

$$\begin{cases} u_{tt} = c^2 u_{xx}, & |x| < \infty, t > 0 \\ u(x, 0) = H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \\ u_t(x, 0) = 0. \end{cases}$$

Here $H(x)$ is called the Heaviside function (or the unit step function), and we'll use the H notation throughout these Notes to designate the Heaviside function. By d'Alembert's formula, $u(x, t) = \frac{1}{2}\{H(x + ct) + H(x - ct)\}$, but this expression is not very informative about the geometry of the solution or its behavior. Also, there are (propagating) discontinuities along the lines $x \pm ct = 0$, so plotting these in x, t space (see figure 3), one sees the domain (upper half of the plane) is separated into 3 regions. Pick any point (x_0, t_0) in the domain, but not on the characteristic lines $x \pm ct = 0$. There are two *characteristics* or *characteristic lines* running through the point, namely $x + ct = x_0 + ct_0$ and $x - ct = x_0 - ct_0$. (To be more precise, we mean the set $\{(x, t) \in \mathbb{R} \times \mathbb{R}^+ : x + ct = x_0 + ct_0\}$ for the first equation, and an analogous set of points for the second expression.) Then d'Alembert's formula states that the solution u at point (x_0, t_0) is given by

$$u(x_0, t_0) = \frac{1}{2}\{f(x_0 + ct_0) + f(x_0 - ct_0)\} + \frac{1}{2c} \int_{x_0 - ct_0}^{x_0 + ct_0} g(y) dy.$$

Therefore, noting that the characteristics through (x_0, t_0) form a triangle, the *characteristic triangle*, with the x -axis, d'Alembert's formula just states that

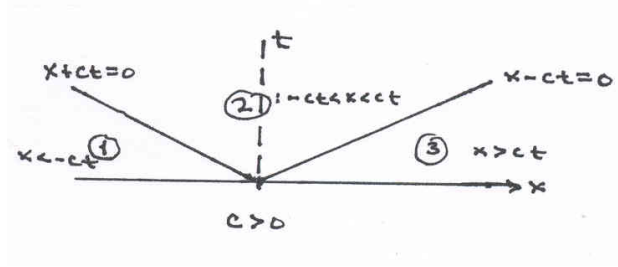


Figure 3: Domain sectioning for example 3

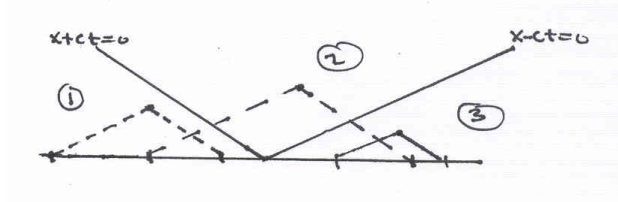


Figure 4: Characteristic triangles for points chosen in the discussion of example 3

$u(x_0, t_0)$ is just the sum of the averages of the initial displacement function f at the endpoints of the base interval of the triangle, $\{f(x_0 + ct_0) + f(x_0 - ct_0)\}/2$, plus the (scaled) integral of the initial velocity function g over the interval. (Actually the last term is just t_0 times the average of g over the interval $[x_0 - ct_0, x_0 + ct_0]$.)

In the example above, where $u = \{f(x + ct) + f(x - ct)\}/2$, for any point in region 1 ($x + ct < 0$), any characteristic triangle will have $f = 0$ at its endpoints, so $u \equiv 0$ in region 1. (see figure 4.) In region 3 ($x - ct > 0$), $f = 1$ at each endpoint of the base of the triangle, so $u \equiv 1$ for any point in region 3. In region 2 ($-ct < x < ct$), one end of the triangle falls in the $x < 0$ where $f = 0$, and the other end falls in $x > 0$, where $f = 1$, so $u \equiv (1 + 0)/2 = 1/2$ in region 2. The piecewise constant nature of the solution comes from the piecewise constant aspect of the initial data.

Remark: Since f in this example does not meet the conditions of the theorem and u is discontinuous across the characteristics $x \pm ct = 0$, then u is not a classical solution to the Cauchy problem. This is an important point. The constructed u satisfies the problem everywhere except on these charac-

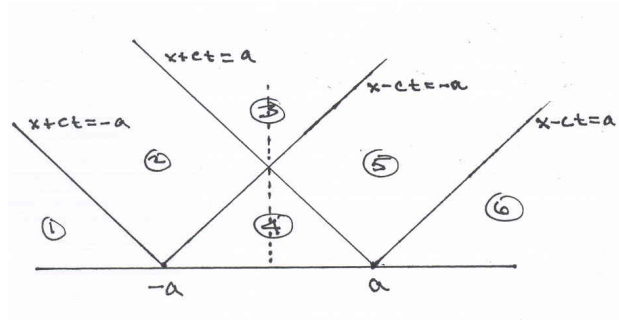


Figure 5: Numbered regions determined by the characteristics for example 4

teristics, so it holds almost everywhere in the upper half plane. But how about on these two lines, and how are we to think about this "solution"? We will have more to say about this situation later, but think of u as a *weak solution* to the IVP. More work is necessary to determine the appropriate value of u on the characteristics associated with the point of discontinuity for $u(x, 0) = f(x) = H(x)$.

Example 4

$$\begin{cases} u_{tt} = c^2 u_{xx}, & |x| < \infty, t > 0 \\ u(x, 0) = hH(a - |x|) & \\ u_t(x, 0) \equiv 0 & |x| < \infty. \end{cases}$$

Here c, a, h are positive constants. Now there are discontinuities at $x = \pm a$, so sketch in the characteristics from these points $(x, t) = (\pm a, 0)$ (figure 5). Pick a point in each of the 6 subregions of the domain and proceed to draw a characteristic triangle with each point chosen at the vertex of the triangle. Then apply d'Alembert's solution formula as interpreted above. You should end up determining the solution to be $u \equiv 0$ in regions 1, 3, 6; $u \equiv h/2$ in regions 2 and 5; and $u \equiv h$ in region 4. Hence, piece-wise constant initial data lead to piece-wise constant solution. Drawing a horizontal line at level $0 < t < a/c$ and for $t > a/c$, you should picture the initial "block" of height h being "pulled down" at its ends as t increases until time $t = a/c$ when the block is uniform height $h/2$, but is twice as wide. Then it divides into two blocks and for larger times the two blocks "run away from each other" along their respective characteristics at speed c . (Figure 6 shows the graph of the solution.)

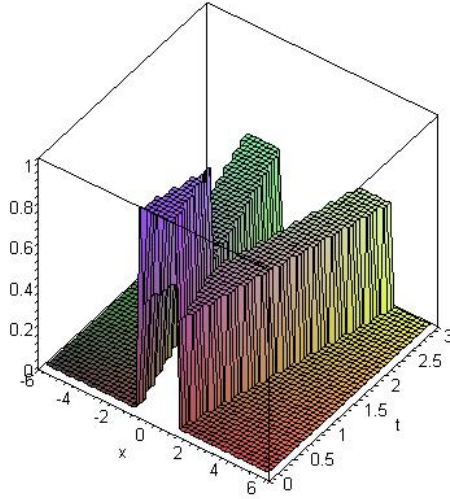


Figure 6: Solution to example 4 with $c = h = a = 1$

Example 5

$$\begin{cases} u_{tt} = u_{xx}, & |x| < \infty, t > 0 \\ u(x, 0) \equiv 0 \\ u_t(x, 0) = AH(\delta - |x - a|) \end{cases}$$

Again $A, a, \delta > 0$ are constants, and $H(\delta - |x - a|) = 1$ if and only if $a - \delta < x < a + \delta$. Now the domain is broken up into 6 regions as shown in figure 7. Then by d'Alembert's formula, we have

$$\begin{aligned} u &\equiv 0 \text{ in regions 1 and 3;} \\ u &= \frac{1}{2} \int_{x-t}^{x+t} A dy = At \text{ in region 2;} \\ u &= \frac{1}{2} \int_{x-\delta}^{x+\delta} A dy = A\delta \text{ in region 5;} \\ u &= \frac{1}{2} \int_{x-\delta}^{x+t} A dy = \frac{A}{2} \{x + t - a + \delta\} \text{ in region 4;} \\ u &= \frac{1}{2} \int_{x-t}^{a+\delta} A dy = \frac{A}{2} \{a + \delta - x + t\} \text{ in region 6.} \end{aligned}$$

Exercises

In each of these exercises, draw the three regions divided by the characteristics emerging from the discontinuity on the x -axis, and determine the solution in each region. For the equation $u_{tt} = u_{xx}$, let

1. $u(x, 0) \equiv 0$ and $u_t(x, 0) = H(x) \cos(x)$.

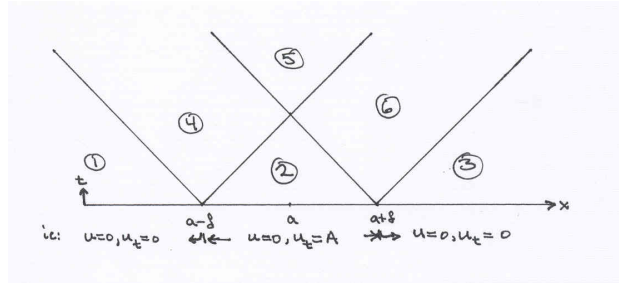


Figure 7: The nature of the domain for example 5

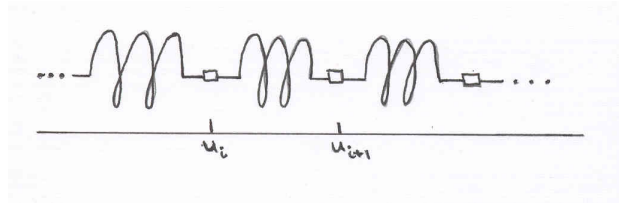


Figure 8: Chain of spring-mass elements for development of the wave equation.

2. $u(x, 0) = 1 + xH(x - 1)$ and $u_t(x, 0) \equiv 0$.

If we number the regions as in figure 4, then for this exercise, initial conditions 1, the answer is $u(x, t) = (0, \sin(x + t)/2, \cos(x) \sin(t))$ in regions (1,2,3), respectively. For the second initial conditions, $u(x, t) = (1, 1 + (x+t)/2, 1+x)$ in regions (1,2,3). (Remember, in this last exercise, be sure to interpret the functional notation correctly; if you have $u = \{f(x + t) + f(x - t)\}/2$ in region 3, e.g., then $f(x + t) = 1 + x + t$.)

Summary: You need to know d'Alembert's formula for the solution of Cauchy's problem. You need to compute characteristics for wave equations and know how to use them to construct a solution to a given problem.

5.4 Appendix: A couple of physical problems leading to a wave equation

Consider a chain of elements, each with mass m , and a spring with spring constant k , of length Δx (see Figure 8). Denote the position of the mass associated with the i th element to be $u_i \in \mathbb{R}$. The force, F_i , necessary to

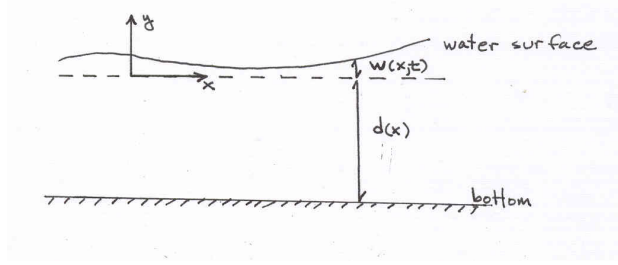


Figure 9: Shallow-water linear-wave notation.

change the original length of the i th spring by amount $\delta_i = u_i - u_{i-1} - \Delta x$ is $F_i = k\delta_i$. Masses are free to move in the x -direction. Their inertia is balanced by the reaction force of the springs. Each element (except the end ones) experiences a spring force from the neighboring $(i-1)$ st and $(i+1)$ st springs, so Newton's law from the i th element is $m \frac{d^2 u_i}{dt^2} = F_{i+1} - F_i = k(u_{i+1} - u_i - u_i + u_{i-1})$, $i = 1, 2, \dots$. Let ρ be the density, so $m = \rho\Delta x$, while the spring constant is described by $\sigma = k\Delta x$. So,

$$\rho\Delta x \frac{d^2 u_i}{dt^2} = \frac{\sigma}{\Delta x}(u_{i+1} - u_i) - \frac{\sigma}{\Delta x}(u_i - u_{i-1}) \quad \text{or} \quad \frac{d^2 u_i}{dt^2} = \frac{\sigma}{\rho} \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}.$$

Letting $\Delta x \rightarrow 0$, then the left-hand side $\rightarrow \partial^2 u / \partial t^2$ and the right-hand side $\rightarrow \frac{\sigma}{\rho} \partial^2 u / \partial x^2$, so we have the 1D wave equation with wave speed $c = \sqrt{\sigma/\rho}$, and u describes the longitudinal waves along the suspended chain of masses. In the context of pressure density perturbations of a compressible fluid, like air, the wave equation describes 1D sound waves (think organ pipes).

Figure 9 depicts long waves in shallow water of constant density ρ and mean water depth $d(x) > 0$. Let the wave height above the undisturbed surface be $w(x, t)$. Assume particle acceleration to be sufficiently small that the pressure p at any point beneath the surface is essentially hydrostatic, so that pressure $p = \rho g(w - y)$, where g is the gravity acceleration constant. Then the accelerating force in the x -direction is proportional to $p_x = \rho g w_x$ and since it is independent of y it is reasonable to write the x component of particle velocity as $v = v(x, t)$. The acceleration of a fluid particle is given by $\frac{dv}{dt} = v_t + v_x \frac{dx}{dt} = v_t + vv_x = -g w_x$, neglecting any viscous forces. From conservation of mass principle for fluid lying between x and $x + \Delta x$, one can obtain $(v(w + d))_x = -w_t$. If we have $d = \text{constant}$, and consider v, w and their derivatives to be small enough to neglect nonlinear terms, then we

have $v_t = -gw_x$ and $-w_{tt} = dv_{xt}$, so that we arrive at the wave equation $w_{tt} = gdw_{xx}$.